INTEGRABILITY OF STRINGS IN $AdS_3 \times S^3$

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INTRODUCTION

- One of the dual pairs in the AdS/CFT correspondence is:

IIB string theory on $AdS_3 \times S^3 \times T^4$ and

$\mathcal{N} = (4,4)$ SCFT on the symmetric product of T^4.

$AdS_3 \times S^3 \times T^4$ arises as the near horizon geometry of D1-D5 system.

- An aspect less explored in this dual pair is that of integrability.

- On can easily show that the sigma model on $AdS_3 \times S^3$ is classically integrable.

It admits an infinite set of non-local charges.

Given a sigma model based on a group manifold

$$\int \mathrm{Tr}(\partial g^{-1} \partial g)$$

Let us define

$$j = -dgg^{-1}$$

One can construct a one parameter family of flat connections

$$J_{\pm} = \frac{j_{\pm}}{1 \mp x}$$

Then the following quantity is independent of time

$$U(x) = P \exp \int_{-\infty}^{\infty} \frac{d\sigma}{2} \left(\frac{j_{+}}{1-x} - \frac{j_{-}}{1+x} \right)$$

This serves as the generating function for the charges.

Since the sigma model on $AdS_3 \times S^3$ is based on the supergroup $SU(1,1|2)$ the above argument works for this case.

- It might be possible to determine the complete spectrum of strings in this system using integrability.

- This can help to understand the states of the symmetric product conformal field theory better.

- One of the well studied black hole background in string theory admits the near horizon geometry $BTZ \times S^3$.

The BTZ geometry is obtained from AdS_3 by discrete identifications.

The integrable structure present in the case of AdS_3 is preserved under these identifications.

- Recall certain features of integrability for the case of $AdS_5 \times S^5$.

Magnon states played an important role in this case.

- Magnons are BPS they obey the dispersion relation

$$E - J = \sqrt{1 + f(\lambda)\sin^2\frac{p}{2}}$$

where:

λ is the t'Hooft coupling,

p is the momentum of the magnon,

E the energy,

J the R-charge.

- Since they are BPS they can be identifed as states both in the field theory and as classical solutions of the string sigma model on $AdS_5 \times S^5$

- Using their supersymmetric properties and integrability it is possible to determine the S-matrix upto a phase.
- The phase is constrained by crossing symmetry.
- There is a conjectured exact expression for the phase. This agrees with the semi-classical expansion of the phase determined by studying scattering of the magnons.

- The D1-D5 system also admits magnons states.
- We wish to determine:
  - The dispersion relation of the magnons.
  - The S-matrix of the magnons.

MAGNONS: DEFINITION

● Boundary CFT of the D1-D5 system: $\mathcal{N} = (4,4)$ CFT on a resolution of the symmetric product orbifold

$$\mathcal{M} = (T^4)^{Q_1 Q_5}/S(Q_1 Q_5).$$

The global part of the $\mathcal{N} = (4,4)$ algebra is the $SU(1,1|2) \times SU(1,1|2)$.
Bosonic part: $(SL(2,R) \times SU(2)) \times (SL(2,R) \times SU(2))$.

● The chiral primaries of the CFT satisfy the condition:

$$L_0 = J^3, \tilde{L}_0 = \tilde{J}^3$$

.

They are the ground states in $\mathbb{Z}_J$ twisted sector of the orbifold with charge: $(\frac{J-1}{2}, \frac{J-1}{2})$

- The subgroup which leaves the chiral primaries invariant:

consists of a left moving

$$SU(1|1) \times SU(1|1)$$

and a right moving:

$$SU(1|1) \times SU(1|1)$$

- Let us focus on the subgroup $SU(1|1) \times SU(1|1)$.

$$\{Q_1, S_1\} = C_1, \qquad \{Q_2, S_2\} = C_2$$
$$\{Q_1, Q_2\} = 0, \qquad \{S_1, S_2\} = 0,$$
$$\{Q_1, S_2\} = 0, \qquad \{S_1, Q_2\} = 0.$$

C_1, C_2 are central elements of the algebra.

- Magnon Excitations are the following excitations above the chiral primary

$$|\phi_{p_1}\phi_{p_2}\cdots\phi_{p_j}\rangle_J \otimes |0\rangle_J = J^-_{p_1} J^-_{p_2}\cdots J^-_{p_j}|0\rangle_J \otimes |0\rangle_J.$$

where

$$J^-_p = \sum_{k=1}^{J} e^{ipk} J^-_{(k)},$$

and $J^-_{(k)}$ is the lowering operator of the left moving $SU(2)$ R-current of the k-th copy of the torus involved in the $\mathbb{Z}_J$ twisted sector.

To satisfy orbifold group invariance condition we need to impose the condition

$$\sum_i p_i = 0.$$

MAGNONS:THE DISPERSION RELATION

- The dispersion relation of these magnons is obtained from the BPS condition as a result of a central extension of $SU(1|1) \times SU(1|1)$
- The centrally extended algebra is

$$
\begin{aligned}
\{Q_1, S_1\} &= C_1, & \{Q_2, S_2\} &= C_2, \\
\{Q_1, Q_2\} &= C_3 - iC_4, & \{S_1, S_2\} &= C_3 + iC_4, \\
\{Q_1, S_2\} &= 0, & \{S_1, Q_2\} &= 0.
\end{aligned}
$$

Extended by more central charges: C_3, C_2.

Consistent with the condition: $Q^\dagger = S$

- The centrally extended algebra is a $\mathcal{N} = 2$ Poincaré algebra in $2+1$ dimensions.

- The BPS condition is given by

$$(C_1 + C_2)^2 = (C_1 - C_2)^2 + 4(C_3^2 + C_4^2)$$

- It can be shown that using
  - Conformal pertubation theory on the symmetric product.
  - Supersymmetric properties of magnons as solitons of the string sigma model

A single magnon is BPS and carries the following values for the central charge

$$C_3 - iC_4 = \alpha(e^{-ip} - 1), \qquad C_3 + iC_4 = \alpha^*(e^{ip} - 1),$$
$$C_1 - C_2 = 1, \qquad C_1 + C_2 = E - J$$

- Substituting the values of the central charges in in the BPS condition we obtain

$$E - J = \sqrt{1 + 16g^2 \sin^2 \frac{p}{2}}$$

where $g = |\alpha|$ is a function of the parameters of the D1-D5 system. The dispersion relation for Q magnons is given by

$$E - J = \sqrt{Q^2 + 16g^2 \sin^2 \frac{p}{2}}$$

- From the first order perturbation theory in the symmetric product orbifold we see

$$16g^2 = \lambda^2 \frac{Q_1 Q_5}{\pi}$$

where λ is the strength of the marginal perturbation which resolves the symmetric product orbifold

- From the dispersion relation of the magnons obtained using the sigma model we see

$$16g^2 = \frac{R^2}{\pi\alpha} = g_6^2 \frac{Q_1 Q_5}{\pi^2}$$

where R is the radius of S^3 and g_6 is the 6d-string coupling.

- A natural identification is $\lambda = g_6$, then the function g remains the same through out the parameter space of the D1-D5 system.

- We will show that this holds true at one loop in the sigma model perturbation theory.

- For further discussion we adopt the following intrinsic parametrization
- Introduce the spectral parameters x^+, x^-:

$$\frac{x^+}{x^-} = e^{ip}$$

subject to the constraint

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = i\frac{Q}{g}$$

Let

$$c = -i(x^+ - x^-)$$

- Then the central charges carried by the magnon are given by

$$C_1 + C_2 = 2gc - 1, \qquad \bar{C} = C_3 - iC_4 = \alpha(\frac{x^-}{x^+} - 1),$$

$$C = C_3 + iC_4 = \alpha^*(\frac{x^+}{x^-} - 1), \qquad C_1 = gc, \qquad C_2 = gc - 1.$$

THE S-MATRIX FOR THE MAGNONS

- The S-matrix for the scattering of two magnons is constrained by symmetries and integrability.
- We need the following facts about $SU(1|1)$ algebra:

$$\{Q_1, S_1\} = C_1.$$

It admits a charge B which obeys

$$[B, Q_1] = -2Q_1, \qquad [B, S_1] = 2S_1.$$

The quadratic Casimir is given by

$$\mathcal{J} = 2[Q_1, S_1] + \{B, C_1\}.$$

- The S-matrix acts on the tensor product of the two magnon Hilbert space

It can be written as

$$\mathcal{S}_{12} = \mathcal{P}_{12}\mathcal{R}_{12}$$

where $\mathcal{P}_{12}$ is the exchange operator and $\mathcal{R}_{12}$ is the R-matrix.

● Invariance under the sum of the $SU(1|1)$ generators of the two magon Hilbert space constraints the R-matrix to the form

$$\mathcal{R}_{12} = X_{12}\mathcal{I}^{(12)} + Y_{12}\mathcal{J}^{(12)}$$

X_{12}, Y_{12} are scalars depending on the spectral parameters of the magnons.

$\mathcal{I}^{(12)}$ is the identity. $\mathcal{J}^{(12)}$ is the quadratic Casimir of the sum of the $SU(1|1)$ generators of the two magon Hilbert space.

● The R-matrix satisfies the unitarity constraint and the Yang-Baxter relations.

$$\mathcal{R}_{12}\mathcal{R}_{21} = \mathcal{I}_{12}, \quad \mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.$$

- Imposing these conditions determines the scalars X_{12} and Y_{12} as

$$X_{12} = \hat{X}_{12}\exp(i\Theta_{(12)}), \qquad Y_{12} = \hat{Y}_{12}\exp(i\Theta_{(12)})$$

$\hat{X}_{12}, \hat{Y}_{12}$ are explicitly known functions of the spectral parameters.

The phase Θ_{12} is not fixed, but satisfies the unitarity constraint

$$\Theta_{12} + \Theta_{21} = 0$$

● To summarize the S-matrix constructed satisfies the following invariance properties under $SU(1|1) \times SU(1|1)$.

$$
\begin{aligned}
[Q_1^{(1)} \otimes 1 + (-1)^F \otimes Q_1^{(2)}, \mathcal{S}_{12}] &= 0, \\
[S_1^{(1)} \otimes 1 + (-1)^F \otimes S_1^{(2)}, \mathcal{S}_{12}] &= 0, \\
[C^{(1)} S_2^{(1)} \otimes C_2^{(2)} + (-1)^F C_2^{(1)} \otimes C^{(2)} S_2^{(2)}, \mathcal{S}_{12}] &= 0, \\
[\bar{C}^{(1)} Q_2^{(1)} \otimes C_2^{(2)} + (-1)^F C_2^{(1)} \otimes \mathcal{C}^{(2)} Q_2^{(2)}, \mathcal{S}_{12}] &= 0.
\end{aligned}
$$

- The action of the S-matrix on the two magnon state can be explictly written as

$$\mathcal{S}_{12}|\phi_{p_1}\phi_{p_2}\rangle \otimes |0\rangle = S_0(x^{\pm}, y^{\pm}) \left(\frac{x^+ - y^-}{x^- - y^+}\right)^2 |\phi_{p_2}\phi_{p_1}\rangle \otimes |0\rangle$$

$x^{\pm}, y^{\pm}$ parametrizes the rapidity parameters of the first magnon and second magnon respectively.

$S_0(x^{\pm}, y^{\pm})$ is the undetermined phase factor.

- We parametrize the phase factor as

$$\begin{aligned} S_0(x^\pm, y^\pm) &= \sigma_{\rm BDS} \times \sigma^2(x^\pm, y^\pm), \\ &= \frac{x^- - y^+}{x^+ - y^-} \frac{1 - \frac{1}{x^+ y^-}}{1 - \frac{1}{x^- y^+}} \times \sigma^2(x^\pm, y^\pm), \end{aligned}$$

where

$$\sigma_{\rm BDS} = \frac{x^- - y^+}{x^+ - y^-} \frac{1 - \frac{1}{x^+ y^-}}{1 - \frac{1}{x^- y^+}}.$$

This is done so that difference in the phase factor from the $\mathcal{N} = 4$ case is captured in the function $\sigma(x^\pm, y^\pm)$.

- To summarize:
  - Symmetries and Integrability constrain both the dispersion relation and the S-matrix.
  - For the case of dispersion relation, the undetermined part lies in the function g and its dependence on the parameters of the D1-D5 sysem.
  - For the case of the S-matrix the undetermined part is the phase factor.
- We will now determined both these to one loop in the sigma model coupling.

MAGNONS AT STRONG COUPLING

- At strong coupling magnons are classical solutions of the string sigma model in $AdS_3 \times S^3$.
- There are 3 interesting limits at strong coupling
  - Plane wave limit:

$$g \to \infty, \quad k = 2gp \text{ fixed}, \quad Q \text{ fixed}$$

The dispersion relation reduces to

$$E - J = \sqrt{Q^2 + k^2}$$

The spetral parameters are real eg. for a single magnon

$$x^+ \sim x^- = r = \frac{1}{k}(1 + \sqrt{1 + k^2})$$

- Giant magnon limit:

$$g \longrightarrow \infty, \qquad , p, Q \text{ fixed}$$

The spectral parameters lie on the unit circle

$$x^{\pm} \sim \frac{1}{x^{-}} \sim \exp\left(i\frac{p}{2}\right)$$

- Dyonic giant magnon limit

$$g \longrightarrow \infty, \quad Q \longrightarrow \infty, \quad \frac{Q}{g}, p \text{ fixed}$$

The spectral parameters in this case are complex and of $O(g^0)$ since

$$x^{+} - x^{-} + \frac{1}{x^{+}} - \frac{1}{x^{-}} = i\frac{Q}{g}$$

- From the dispersion relations obtained in these limits it is seen

$$4g = \frac{R^2}{\pi\alpha'}, \quad \text{for} \quad g \to \infty$$

- Possible corrections: organized in sigma model perturbation

$$4g = \frac{R^2}{\pi\alpha'}, + g_0 + O((\frac{R^2}{\pi\alpha'})^{-1})$$

substituting this in the exact form for the dispersion relation we obtain

$$E - J = \frac{R^2}{\pi\alpha'}\left|\sin\frac{p}{2}\right| + g_0\left|\sin\frac{p}{2}\right| + O(\frac{R^2}{\pi\alpha'})^{-1} + \cdots .$$

- We will show that the one loop term g_0 vanishes.

- We have parametrized the undetermined phase factor in terms of the function

$$\sigma(x^\pm, y^\pm) = e^{i\theta(x^\pm, y^\pm)}$$

- The phase factor admits a semi-classical expansion

$$\theta(x^\pm, y^\pm) = g\left(\theta_0(x^\pm, y^\pm) + \frac{1}{g}\theta_1(x^\pm, y^\pm) + \frac{1}{g^2}\theta_2(x^\pm, y^\pm) + \cdots\right).$$

THE LEADING PHASE FACTOR

- The leading semi-classical phase shift $\theta_0(x^{\pm}, y^{\pm})$ can be evaluated from the two magnon classical solution.

- Compare the solution at $t \to \pm\infty$ to the single solition solution.

Evaluate the time delay for ΔT_{12} relative to free propagation.

Then the semi-classical phase shift is given by

$$\frac{\partial \theta_0(p_1, p_2)}{\partial E_{p_1}} = \Delta T_{12}$$

- The time delay just depends on classical solution.

The classical solution is a string configuration in $R \times S^3$.

Thus the time delay must be the same as for the case of $AdS_5 \times S^5$ for which the magnons are oriented along a S^3 in S^5.

- This implies that the semi-classical phase shift must be the same as that evaluated for the case of $AdS_5 \times S^5$

$$\theta_0(x^{\pm}, y^{\pm}) = k(x^+, y^+) - k(x^+, y^-) - k(x^-, y^+) + k(x^-, y^-),$$

where

$$k(x, y) = \left[\left(y + \frac{1}{y}\right) - \left(x + \frac{1}{x}\right)\right] \log\left(1 - \frac{1}{xy}\right).$$

● As a simple check we can explicitly evaluate the phase shift for the magnons in $R \times S^3$ in the giant magnon limit and compare with the above expression.

We obtain agreement.

ONE-LOOP CORRECTIONS

● One-loop corrections to the dispersion relation and to the phase shift are evaluated by

$$\Delta E(p) = \frac{1}{2\pi}\sum_{I=1}^{N_F}(-1)^{F_I}\int_{-\infty}^{\infty} dk \frac{\partial \delta_I(k;p)}{\partial k}\sqrt{k^2+1}.$$

$$\begin{aligned} 2\theta_1(p_1,p_2) \;&=\; \frac{1}{4\pi}\sum_{I=1}^{N_F}(-1)^{F_I}\int_{-\infty}^{\infty} dk \left(\frac{\partial \delta_I(k;p_1)}{\partial k}\delta_I(k,p_2)\right. \\ &\qquad \left. -\frac{\partial \delta_I(k;p_2)}{\partial k}\delta_I(k,p_1)\right). \end{aligned}$$

$\delta(k;p)$ be the phase shift corresponding to the scattering of a plane wave off the either a giant dyonic magnon with momentum p.

k is the wave number of the plane wave. I labels the fluctuations of the magnons with Bose/Fermi statistics depending on the sign of $(-1)^{F_I}$.

● Thus the problem is reduced to the evaluation of the phase shifts for various plane wave fluctuations about a background magnon.

● This can be evaluated by the strategy: Consider a $n+1$ giant dyonic magnon solution.

Take a plane wave limit on one of them

Then this magnon becomes the plane wave fluctuation.

● Let us parametrize S^3 as

$$\{(Z_1, Z_2) : |Z_1|^2 + |Z_2|^2 = 1\} \quad \leftrightarrow \quad g = \begin{pmatrix} Z_1 & -iZ_2 \\ -i\bar{Z}_2 & \bar{Z}_1 \end{pmatrix} \in SU(2),$$

Then the plane wave phase shifts are

$$
\begin{aligned}
\delta_{Z_1} &= 0, \\
\delta_{\bar{Z}_1}(r) &= 0, \\
\delta_{Z_2} &= -\left[2i \ln\left(\frac{r-x^+}{r-x^-}\right) - i \ln\left(\frac{x^+}{x^-}\right)\right]. \\
&= -2G \\
\delta_{\bar{Z}_2}(1/r) &= \left[2i \ln\left(\frac{r-x^+}{r-x^-}\right) - i \ln\left(\frac{x^+}{x^-}\right)\right] \\
&= 2G
\end{aligned}
$$

r is the spectral parameter of the plane wave and is related to its wave number and frequency as

$$
k = \frac{2r}{r^2-1}, \qquad \omega = \frac{r^2+1}{r^2-1}
$$

• For coordinates in AdS_3, since they couple trivially, fluctuations along these do not suffer phase shifts. Indeed

$$\delta_{Y_1} = \delta_{\bar{Y}_1} = \delta Y_2 = \delta_{\bar{Y}_2} = 0$$

• Fermionic coordinates of S^3 and AdS_3 are coupled due to the presence of RR fluxes.

Let the superpartners of the coordinates Z_2 and Y_2 be the complex fermions θ and η respectively. The the phase shifts along these directions are

$$\delta_\theta = \delta_\eta = -G(r, x^\pm), \qquad \delta_{\bar{\theta}} = \delta_{\bar{\eta}} = G(1/r, x^\pm)$$

These relations are obtained using the properties of the sigma model supergroup $SU(1,1|2)$.

● Substituting these values for the phase shifts in the expression for the one loop energy

$$
\begin{aligned}
2\pi\Delta E(x^{\pm}) &= \int_{-1}^{1} dr\sqrt{k(r)^2+m^2}\,\frac{\partial}{\partial r}\left[\delta_{Z_2}+\delta_{\bar{Z}_2}-(\delta_\theta+\delta_\eta+\delta_{\bar{\theta}}+\delta_{\bar{\eta}})\right], \\
&= \int_{-1}^{1} dr\sqrt{k(r)^2+m^2}\,\frac{\partial}{\partial r}\Big[-2G(r;x^{\pm})+2G(1/r;x^{\pm}) \\
&\qquad -2G(1/r,x^{\pm})+2G(1/r,x^{\pm})\Big], \\
&= 0.
\end{aligned}
$$

Thus we conclude that the formula

$$4g=\frac{R^2}{\pi\alpha'}$$

is exact to one loop.

- The one loop correction to the scattering phase:

$$
\begin{aligned}
& 2\theta_1(x^\pm, y^\pm) \\
= \; & \frac{1}{2\pi}\left[\int_{-1}^{+1} dr\, \frac{\partial G(r, x^\pm)}{\partial r} G(r, y^\pm) + \int_{-1}^{+1} dr\, \frac{\partial G(\frac{1}{r}, x^\pm)}{\partial r} G(\frac{1}{r}, y^\pm)\right] \\
& -(x^\pm \leftrightarrow y^\pm),
\end{aligned}
$$

This can be arranged as

$$
2\theta_1(x^\pm, y^\pm) = \chi_1(x^+, y^+) - \chi_1(x^+, y^-) - \chi_1(x^-, y^+) + \chi_1(x^-, y^-),
$$

and $\chi(x, y)$ has an expression in terms of dilogarithms.

$$
\chi_1(x, y) \;=\; -\frac{1}{2\pi}\left[I_1(x, y) - I_1(y, x) + I_2(x, y) - I_2(y, x)\right].
$$

$$
\begin{aligned}
I_1(x,y) &= \left(\ln(x-y) - \frac{1}{2}\ln(y)\right)\ln\left(\frac{x-1}{x+1}\right) \\
&\quad + \left[\mathrm{Li}_2\left(\frac{x+1}{x-y}\right) - \mathrm{Li}_2\left(\frac{x-1}{x-y}\right)\right], \\
I_2(x,y) &= \left(\ln\left(1-\frac{y}{x}\right) - \frac{1}{2}\ln(y)\right)\ln\left(\frac{1-x}{1+x}\right) \\
&\quad + \left[\mathrm{Li}_2\left[(x+1)\frac{y}{y-x}\right] - \mathrm{Li}_2\left[(x-1)\frac{y}{x-y}\right]\right].
\end{aligned}
$$

CROSSING SYMMETRY AND UNITARITY CONSTRAINTS

● Crossing symmetry relates the scattering matrix of a particle to that of its anti-particle.

$$\mathcal{C}^{-1} \otimes I \mathcal{S}_{12}^{T_1}(-p_1, p_2)\mathcal{C} \otimes I \mathcal{S}_{12}(p_1, p_2) = I,$$
$$I \otimes \mathcal{C}^{-1} \mathcal{S}_{12}^{T_2}(p_1, -p_2) I \otimes \mathcal{C}\mathcal{S}_{12}(p_1, p_2) = I.$$

$\mathcal{C}$ is the charge conjugation operator. This implies the following equations for the phase factor

$$S_0(x, y) S_0(\frac{1}{x}, y) = f(x, y)$$
$$S_0(x, y, S_0(x, \frac{1}{y}) = f(x, y)$$

Unitarity then implies

$$f(x,y)f(y,\frac{1}{x})=1$$

- Crossing symmetry and unitarity leads to the following constraint on the phase factor

$$\ln\frac{y^+}{y^-}+i\theta(x,y)-\ln\frac{x^+}{x^-}-i\theta\left(\frac{1}{x},\frac{1}{y}\right)=0.$$

- We can substitute the expansion

$$\theta(x,y) \;\; = \;\; g\theta_0(x,y)+\theta_1(x,y)+\cdots\,.$$

and note that $\theta_0(x,y)$ satisfies

$$g\theta_0\left(\frac{1}{x},\frac{1}{y}\right) = g\theta_0(x,y) + i\ln\left(\frac{x^+}{x^-}\right) - i\ln\left(\frac{y^+}{y^-}\right).$$

and

$$\theta_1\left(\frac{1}{x},\frac{1}{y}\right) = \theta_1(x,y).$$

- This results in the fact that the contraint following from crossing symmetry and unitarity is statisfied to one loop.

SUMMARY

- We used symmetries and integrability to constrain the form of the dispersion relations and the S-matrix for the magnons in $AdS_3 \times S^3$.
- We obtained information about the dependence of the coupling g on the parameters of the D1-D5 system and the phase factor to one loop using semi-classical methods.